

# DIMENSIONS OF SOME LOCALLY ANALYTIC REPRESENTATIONS

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ABSTRACT. Let  $G$  be the group of points of a split reductive group over a finite extension of  $\mathbb{Q}_p$ . In this paper, we compute the dimensions of certain classes of locally analytic  $G$ -representations. This includes principal series representations and certain representations coming from homogeneous line bundles on  $p$ -adic symmetric spaces. As an application, we compute the dimensions in Colmez' unitary principal series of  $\mathrm{GL}_2(\mathbb{Q}_p)$ .

## CONTENTS

1.	Introduction	1
2.	Grade and dimension	3
3.	Arens-Michael envelopes and faithful flatness	4
4.	From $D(\mathfrak{g}, P)$ -modules to $D(G)$ -modules	12
5.	Highest weight modules and dimension	17
6.	Application to equivariant line bundles on Drinfeld's upper half space	22
7.	Dimensions in the unitary principal series for $\mathrm{GL}_2(\mathbb{Q}_p)$	24
	References	26

## 1. INTRODUCTION

Let  $L$  be a finite extension of  $\mathbb{Q}_p$  and let  $G = \mathbf{G}(L)$  be the group of  $L$ -valued points of a split connected reductive algebraic group  $\mathbf{G}$  over  $L$ . Let  $P \subseteq G$  be a parabolic subgroup.

Admissible Banach space representations or locally analytic representations of  $G$  admit a well-behaved notion of (canonical) dimension. The rational representations coming from the algebraic group  $\mathbf{G}$  or the traditional smooth representation from Langlands theory are known to have dimension zero. Moreover, any representation which is not zero-dimensional has dimension greater or equal to half the dimension of the minimal nilpotent orbit of  $\mathbf{G}$  [2], [31]. Besides these general results, the dimensions of even very explicit representations like principal series representations have not been computed so far. In this paper, we make an attempt to close this gap and determine the dimensions of certain families of representations. This includes principal series representations as well as

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representations coming from  $p$ -adic symmetric spaces. The technical key result is that the functor  $\mathcal{F}_P^G$  introduced by Orlik and the second author in [26] from Lie algebra representations of  $\mathfrak{g}$  endowed with a compatible action of  $P$  to locally analytic  $G$ -representations preserves the dimension.

As an application, we compute the dimensions in Colmez' unitary principal series of  $\mathrm{GL}_2(\mathbb{Q}_p)$  [9]. Let  $\Pi(V)$  denote the unitary representation associated by Colmez'  $p$ -adic local Langlands correspondence [10] to an absolutely irreducible 2-dimensional  $p$ -adic Galois representations  $V$  of  $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ . The resulting map

$$V \mapsto \dim \Pi(V)$$

is bounded above by the number 2 due to the presence of infinitesimal characters. We show that its restriction to trianguline representations is constant with value 1. This raises the question whether there are absolutely irreducible 2-dimensional representations  $V$  of  $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  with the property  $\dim \Pi(V) = 2$ .

In the following we give more details on the individual sections of this paper. In section 2 we review basic notions of dimension theory and establish two auxiliary lemmas. In section 3 we develop a framework which allows us to prove faithful flatness of Arens-Michael envelopes in many situations. In section 4 we combine this result, in the case of the universal enveloping algebra  $U(\mathfrak{g})$  of  $\mathfrak{g} = \mathrm{Lie}(G)$ , with a study of the functor  $\mathcal{F}_P^G$  and prove that the latter preserves dimensions. On the level of Lie algebra representations, canonical dimension coincides with the more traditional Gelfand-Kirillov dimension and this enables us to give explicit dimension formulas for the representations  $\mathcal{F}_P^G(M)$  whenever the Gelfand-Kirillov dimension for  $M$  (viewed as an  $U(\mathfrak{g})$ -module) is known. We illustrate this in section 5 in the case of the classical parabolic Bernstein-Gelfand-Gelfand category for  $\mathfrak{p} \subseteq \mathfrak{g}$  where  $\mathfrak{p} = \mathrm{Lie}(P)$ . For example, the dimension of the locally analytic parabolic induction  $\mathrm{Ind}_P^G(V)$  where  $V$  is a locally analytic  $P$ -representations on a finite-dimensional vector space equals the vector space dimension of  $\mathfrak{g}/\mathfrak{p}$ . We also remark that the dimensions of irreducible objects in the  $BGG$ -category can be computed out of the Kazhdan-Lusztig conjecture through Joseph's Goldie rank polynomials. The main result of [26] shows that the functor  $\mathcal{F}_P^G$  preserves irreducibility in many cases which yields the dimensions of all the irreducible  $G$ -representations which can be constructed through a functor of type  $\mathcal{F}_P^G$ . In section 6 we let  $G = \mathrm{GL}_{d+1}(L)$  and compute the dimension of locally analytic representations coming from homogeneous line bundles on Drinfeld's upper half space [25]. In section 7 we give the aforementioned application to  $\mathrm{GL}_2(\mathbb{Q}_p)$  and its unitary principal series.

*Notation and conventions:* We denote by  $p$  a prime number and consider fields  $L \subset K$  which are both finite extensions of  $\mathbb{Q}_p$ . Let  $o_L$  and  $o_K$  be the rings of integers of  $L$ , resp.  $K$ , and let  $|\cdot|_K$  be the absolute value on  $K$  such that  $|p|_K = p^{-1}$ . The field  $L$  is our "base field", whereas we consider  $K$  as our "coefficient field". For a locally convex

$K$ -vector space  $V$  we denote by  $V'_b$  its strong dual, i.e., the  $K$ -vector space of continuous linear forms equipped with the strong topology of bounded convergence. Sometimes, in particular when  $V$  is finite-dimensional, we simplify notation and write  $V'$  instead of  $V'_b$ . All finite-dimensional  $K$ -vector spaces are equipped with the unique Hausdorff locally convex topology.

We let  $\mathbf{G}$  be a split reductive group scheme over  $o_L$  and  $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$  a maximal split torus and a Borel subgroup scheme, respectively. We denote the base change to  $L$  of these group schemes by the same letters. We let  $\mathbf{B} \subseteq \mathbf{P}$  be a parabolic subgroup and let  $\mathbf{L_P}$  the unique Levi subgroup which contains  $\mathbf{T}$ . By  $G_0 = \mathbf{G}(o_L)$ ,  $B = \mathbf{B}(o_L)$ , etc., and  $G = \mathbf{G}(L)$ ,  $B = \mathbf{B}(L)$ , etc., we denote the corresponding groups of  $o_L$ -valued points and  $L$ -valued points, respectively. Finally, Gothic letters  $\mathfrak{g}$ ,  $\mathfrak{p}$ , etc., will denote the Lie algebras of  $\mathbf{G}$ ,  $\mathbf{P}$ , etc.:  $\mathfrak{g} = \text{Lie}(\mathbf{G})$ ,  $\mathfrak{t} = \text{Lie}(\mathbf{T})$ ,  $\mathfrak{b} = \text{Lie}(\mathbf{B})$ ,  $\mathfrak{p} = \text{Lie}(\mathbf{P})$ ,  $\mathfrak{l_P} = \text{Lie}(\mathbf{L_P})$ , etc.. Base change to  $K$  is usually denoted by the subscript  $K$ , for instance,  $\mathfrak{g}_K = \mathfrak{g} \otimes_L K$ .

## 2. GRADE AND DIMENSION

In this section we introduce some basic notions in dimension theory and establish two simple lemmas. The term module always means *left* module. Noetherian rings are two-sided noetherian and other ring-theoretic properties are used similarly.

We recall the notion of an Auslander regular ring [21]. Let  $R$  be an arbitrary associative unital ring. For any  $R$ -module  $N$  the *grade*  $j_R(N)$  is defined to be either the smallest integer  $k$  such that  $\text{Ext}_R^k(N, R) \neq 0$  or  $\infty$ . Now suppose that  $R$  is (left and right) noetherian. If  $N \neq 0$  is finitely generated, then its grade  $j_R(N)$  is bounded above by the projective dimension of  $N$ . A noetherian ring  $R$  is called *Auslander regular* if its global dimension is finite and if every finitely generated  $R$ -module  $N$  satisfies *Auslander's condition*: for any  $k \geq 0$  and any  $R$ -submodule  $L \subseteq \text{Ext}_R^k(N, R)$  one has  $j_R(L) \geq k$ .

Let  $R$  be an Auslander regular ring of finite global dimension  $\text{gld}(R)$  and  $M$  an  $R$ -module. The number

$$\dim_R M := \text{gld}(R) - j_R(M)$$

is called the *canonical dimension* of  $M$ . One has

$$\dim_R M = \max\{\dim_R M', \dim_R M''\}$$

for an exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ . Moreover,  $\dim_R 0 = -\infty$ .

Let  $\tau$  be an automorphism of  $R$  and let  $M$  be a left  $R$ -module. We denote by  ${}^\tau M$  the abelian group  $M$  with the left  $R$ -action  $r.m := \tau(r)m$  and call  ${}^\tau M$  the *twist of  $M$  with  $\tau$* . In case of a right module  $M$  we denote the analogous construction by  $M^\tau$ .

**Lemma 2.1.** *Twisting with  $\tau$  has the following properties:*

- (i) *the functor  $M \rightarrow {}^\tau M$  is an auto-equivalence on the category of all  $R$ -modules,*

- (ii)  $M$  is finitely generated if and only if  $\tau M$  is finitely generated,
- (iii) there are canonical isomorphisms  $\text{Ext}_R^k(\tau M, R) \simeq \text{Ext}_R^k(M, R)^\tau$  for all  $k$ ,
- (iv) one has  $j_R(M) = j_R(\tau M)$ .

*Proof.* Twisting with  $\tau^{-1}$  yields a quasi-inverse, so (i) is clear. (ii) is trivial so let us turn to (iii). In case of  $k = 0$  the isomorphism is given explicitly by sending a linear form  $f$  on  $\tau M$  to the linear form  $\tau \circ f$  on  $M$ . According to (i) a projective resolution  $P_\bullet$  for  $M$  yields a projective resolution  $\tau P$  for  $\tau M$ . Since the isomorphism for  $k = 0$  is natural in  $M$ , we are done. (iv) follows formally from (iii).  $\square$

**Lemma 2.2.** *Let  $R \rightarrow S$  be a faithfully flat ring extension between noetherian rings. Let  $M$  be a finitely generated  $R$ -module and put  $M_S := S \otimes_R M$ . We have*

$$j_S(M_S) = j_R(M) .$$

*Proof.* We have  $\text{Ext}_R^k(M, R) \otimes_R S \simeq \text{Ext}_S^k(M_S, S)$  for all  $k$ . Indeed, since  $R \rightarrow S$  is flat, choosing a free resolution of  $M$  by finitely generated free modules reduces us to the case  $k = 0$  and  $M = R$  where the statement is obvious. By faithful flatness of  $R \rightarrow S$ , we have  $\text{Ext}_R^{j_R(M)}(M, R) \otimes_R S \neq 0$  which implies the claim.  $\square$

### 3. ARENS-MICHAEL ENVELOPES AND FAITHFUL FLATNESS

Let  $R$  be a complete discrete valuation ring with field of fractions  $K$  and uniformizer  $\pi$ . Let  $A$  be an  $R$ -algebra, flat as an  $R$ -module, equipped with an increasing and exhaustive filtration

$$F_0 A \subseteq F_1 A \subseteq F_2 A \subseteq \dots$$

by  $R$ -submodules such that  $1 \in F_0 A$  and  $F_i A \cdot F_j A \subseteq F_{i+j} A$  for all  $i, j$ . In particular,  $F_0 A$  is an  $R$ -subalgebra of  $A$ . We make the following three assumptions on this filtration.

- (1) We have  $F_i A \cdot F_j A = F_j A \cdot F_i A$  as  $R$ -submodules of  $A$  for all  $i, j$ ;
- (2) the ring  $F_0 A$  is a commutative noetherian integral domain such that  $F_0 A / \pi F_0 A$  is a regular integral domain;
- (3) the associated graded ring  $gr_\bullet^F A$  is commutative and isomorphic to a polynomial ring over  $F_0 A$  in finitely many, say  $r$ , variables (where the polynomial ring has its usual positive grading by total degree with the variables placed in degree one).

The regularity assumption in (2) means that all local rings of  $F_0 A / \pi F_0 A$  at prime ideals are regular or, equivalently, that the ring  $F_0 A / \pi F_0 A$  has finite global dimension. Of course, any filtration with  $F_0 A = R$  satisfies (2), but there is no point in restricting to this special case at the moment.

Positively filtered algebras  $A$  that satisfy these requirements abound. The main examples we have in mind are universal enveloping algebras of Lie algebras as well as the rings of

(crystalline) differential operators on certain smooth affine  $R$ -schemes. We will give more details at the end of this section.

In the following we will assume that these conditions hold. We then have the  $K$ -algebras

$$(F_0A)_K := F_0A \otimes_R K \quad \text{and} \quad A_K := A \otimes_R K.$$

The algebra  $(F_0A)_K$  has a natural structure of normed algebra by declaring the lattice  $F_0A$  to be the unit ball. We give  $A_K$  the finest locally convex topology making the inclusion map  $(F_0A)_K \hookrightarrow A_K$  continuous (where the source has its norm topology). Our aim in this subsection is to analyze the algebraic and homological properties of the Arens-Michael envelope  $\hat{A}_K$  of the locally convex algebra  $A_K$ . Recall [13]<sup>1</sup> that

$\hat{A}_K := (\text{Hausdorff})$  completion of  $A_K$  w. r. t. all continuous submultiplicative seminorms.

Among our main results will be that  $\hat{A}_K$  is a Fréchet-Stein algebra in the sense of [35] and that the canonical completion homomorphism  $A_K \rightarrow \hat{A}_K$  is a faithfully flat ring extension. As we will see, these results make the homological algebra of  $\hat{A}_K$  quite transparent.

As a first step we will obtain a more accessible description of  $\hat{A}_K$ . To this end, we consider the Rees ring

$$R_\bullet^F(A) := \bigoplus_{i \geq 0} (F_i A) X^i$$

of the filtered ring  $A$ , viewed as a subring of the polynomial ring  $A[X]$ . The ring  $R_\bullet^F(A)$  is noetherian according to [21, II.2.2.1]. For each number  $n \geq 0$  we let  $A_n$  be the image of  $R_\bullet^F(A)$  under the evaluation homomorphism  $A[X] \rightarrow A$  given by  $X \mapsto \pi^n$ . Obviously,  $A_{n+1} \subseteq A_n$  and  $A_0 = A$ . Let  $\hat{A}_n$  be the  $\pi$ -adic completion of  $A_n$  and put  $\hat{A}_{n,K} := \hat{A}_n \otimes_R K$ . All rings  $A_n$ ,  $\hat{A}_n$  and  $\hat{A}_{n,K}$  are noetherian.

In the following we will need some basic results on the interplay between the positive filtration  $F_\bullet A$  on  $A$ , the  $\pi$ -adic filtration on  $A$  and the rings  $A_n$ . Such results are established by K. Ardakov and S. Wadsley in [2] and to be completely clear, we therefore relate our situation to the terminology used in loc.cit. The positively filtered ring  $A$  is an almost commutative  $R$ -algebra in the sense of the definition [2, 3.4]. Moreover, it is deformable and  $A_n$  is its  $n$ -th deformation [2, 3.5]. According to [2, Prop. 3.8] the algebra  $\hat{A}_{n,K}$  is therefore an almost commutative affinoid  $K$ -algebra in the sense of the definition [2, 3.8]. In particular,  $\hat{A}_{n,K}$  is a complete doubly filtered  $K$ -algebra with slice  $\hat{A}_n/\pi\hat{A}_n$  [2, 3.1]. Of course, we have  $A_n/\pi A_n = \hat{A}_n/\pi\hat{A}_n$ .

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<sup>1</sup>The classical notion of Arens-Michael envelope [13] is given for complex algebras, but the definition extends readily to any valued field.

Each ring  $A_n$  has its induced filtration  $F_m A_n := A_n \cap F_m A$ . Since  $gr_{\bullet}^F A$  is flat over  $R$  one has

$$(3.1) \quad F_m A_n = \sum_{i=0, \dots, m} \pi^{in} F_i A .$$

In particular,  $F_0 A_n = F_0 A$ . The graded ring  $gr_{\bullet}^F A_n$  is in fact isomorphic to the graded ring  $gr_{\bullet}^F A$  via the map given on the  $i$ -th homogeneous component as

$$(3.2) \quad F_i A / F_{i-1} A \longrightarrow F_i A_n / F_{i-1} A_n, \quad x + F_{i-1} A \mapsto \pi^{in} x + F_{i-1} A_n ,$$

[2, Lem. 3.5]. In particular,  $gr_{\bullet}^F A_n$  is isomorphic, as a graded ring, to a polynomial ring in  $r$  variables over  $F_0 A_n$ .

The slice  $A_n / \pi A_n$  has the quotient filtration coming from  $F_{\bullet} A_n$ . We let  $gr_{\bullet}^F(A_n / \pi A_n)$  be the associated graded ring. According to [2, Lem. 3.7] the map  $A_n \rightarrow A_n / \pi A_n$  induces an isomorphism of graded rings

$$(3.3) \quad gr_{\bullet}^F A_n / \pi gr_{\bullet}^F A_n \xrightarrow{\cong} gr_{\bullet}^F(A_n / \pi A_n) .$$

Following [2, 3.1] we finally abbreviate

$$\mathrm{Gr}(\hat{A}_{n,K}) := gr_{\bullet}^F(A_n / \pi A_n) .$$

This is a polynomial ring over  $F_0 A_n / \pi F_0 A_n$  in  $r$  variables and is therefore a noetherian regular integral domain according to (2).

**Proposition 3.4.** *The homomorphism  $\hat{A}_{n+1,K} \rightarrow \hat{A}_{n,K}$  is flat for all  $n$ .*

*Proof.* We follow an overall strategy of Berthelot [5, 3.5.3] which is made explicit in [12, 5.3.10]. As a starting point, we equip the ring  $A_n$  with the following 'augmented' filtration:

$$F'_m A_n := A_{n+1} \cdot F_m A_n$$

for all  $m$ . We claim that this filtration satisfies

$$F'_k A_n \cdot F'_\ell A_n \subseteq F'_{k+\ell} A_n$$

for all  $k, \ell$  so that we have an associated graded ring  $gr_{\bullet}^{F'} A_n$ . To prove the claim, it suffices to verify

$$A_{n+1} \cdot F_m A_n = F_m A_n \cdot A_{n+1} .$$

Because of  $A_{n+1} = \sum_{j \geq 0} \pi^{(n+1)j} F_j A$  together with (3.1) this reduces to

$$\pi^{(n+1)j} F_j A \cdot \pi^{in} F_i A = \pi^{in} F_i A \cdot \pi^{(n+1)j} F_j A$$

for each  $i, j$ . However, this is a direct consequence of our hypothesis (1). Secondly, we observe that  $F_0 A_n = F_0 A$  which implies  $gr_0^{F'} A_n = F'_0 A_n = A_{n+1}$ . Finally, we claim that the ring  $gr_{\bullet}^{F'} A_n$  is finitely generated over  $gr_0^{F'} A_n$  by central elements. To start with, the composite

$$F_{m+1} A_n \subseteq F'_{m+1} A_n \rightarrow F'_{m+1} A_n / F'_m A_n$$

is surjective and factors through  $F_{m+1} A_n / F_m A_n$  for all  $m \geq 0$ . We obtain a graded ring homomorphism

$$f : gr_{\bullet}^F A_n \rightarrow gr_{\bullet}^{F'} A_n$$

whose image equals  $F_0 A_n \oplus (\oplus_{m > 0} gr_m^{F'} A_n)$  with  $F_0 A_n \subset gr_0^{F'} A_n$ . According to the isomorphism (3.2) and our hypothesis (3) on  $gr_{\bullet}^F A$ , the source of  $f$  is a polynomial ring over  $F_0 A_n$  in finitely many variables, say  $y_1, \dots, y_r \in gr_1^F A_n$ . It therefore suffices to see that the images of these generators in  $gr_{\bullet}^{F'} A_n$  are central, that is, they commute with  $gr_0^{F'} A_n = F'_0 A_n = A_{n+1}$ . To this end, we choose elements  $x_1, \dots, x_r$  in  $F_1 A$  such that  $y_i = \pi^n x_i + F_0 A_n$ . This is possible according to (3.2). The commutator  $[gr_0^{F'} A_n, f(y_i)]$  vanishes in  $gr_{\bullet}^{F'} A_n$ , if we can show the inclusion  $[A_{n+1}, \pi^n x_i + F_0 A_n] \subseteq A_{n+1}$  inside  $A_n$ . Since  $F_0 A_n = F_0 A_{n+1} \subset A_{n+1}$  and since  $[\cdot, \pi^n x_i]$  is additive, we are reduced to show

$$[\pi^{(n+1)j} z, \pi^n x_i] \in A_{n+1}$$

for any  $z \in F_j A$  and  $j \geq 0$ . Since  $gr_{\bullet}^F A$  is commutative, the commutator  $[z, x_i] \in F_{j+1} A$  lies in fact in the subgroup  $F_j A$ . This implies

$$[\pi^{(n+1)j} z, \pi^n x_i] = \pi^{(n+1)j+n} [z, x_i] \in \pi^n \cdot \pi^{(n+1)j} F_j A \subset \pi^n \cdot F_j A_{n+1} \subset F_j A_{n+1}$$

which proves the claim. All in all, we have now verified the conditions (i),(ii),(iii) appearing in [12, Lem. 5.3.9] for the augmented filtration  $F'_{\bullet} A_n$  and its subring  $F'_0 A_n = A_{n+1}$ . Hence, [12, Prop. 5.3.10] implies the flatness of  $\hat{A}_{n+1, K} \rightarrow \hat{A}_{n, K}$ .  $\square$

The proposition implies that the projective limit

$$\varprojlim_n \hat{A}_{n, K},$$

with its projective limit topology, is a Fréchet-Stein algebra in the sense of [35].

**Proposition 3.5.** *The canonical map  $\varprojlim_n \hat{A}_{n, K} \xrightarrow{\cong} \hat{A}_K$  is an isomorphism of topological algebras.*



*Proof.* Clearly, any  $A_n$  gives rise to a continuous submultiplicative seminorm, say  $||\cdot||_n$ , on  $A_K$  and it suffices to see that these are cofinal in the directed set of all such seminorms on  $A_K$ . According to [2, Lem. 3.1], the graded ring of  $A_n$  relative to its  $\pi$ -adic filtration is isomorphic to a polynomial ring  $(A_n/\pi A_n)[t]$  in one variable  $t$  over  $A_n/\pi A_n$ . Since  $\text{Gr}(\hat{A}_{n,K})$  is an integral domain, the rings  $A_n/\pi A_n$  and  $(A_n/\pi A_n)[t]$  are integral domains, too. This implies that  $||\cdot||_n$  is in fact multiplicative. After these preliminaries, we consider an arbitrary continuous and submultiplicative seminorm  $||\cdot||$  on  $A_K$ . Choose a graded isomorphism between  $gr_{\bullet}^F A$  and a polynomial ring over  $F_0 A$  and lift the variables to elements  $x_1, \dots, x_r$  in  $F_1 A$ . By (2) the ring  $F_0 A$  is an integral domain and, hence, so is  $gr_{\bullet}^F A$ . In particular, the principal symbol map for  $gr_{\bullet}^F A$  is multiplicative. It follows that the ordered monomials  $\underline{x}^{\underline{k}} := x_1^{k_1} \cdots x_r^{k_r}$  for  $\underline{k} := (k_1, \dots, k_r) \in \mathbb{N}^r$  form a basis of the  $F_0 A$ -module  $A$ . Take an element  $a \in A_K$  and write

$$a = \sum_{\underline{k}} a_{\underline{k}} \underline{x}^{\underline{k}}$$

with uniquely determined  $a_{\underline{k}} \in (F_0 A)_K$ . Let  $|\cdot|$  be the norm on  $(F_0 A)_K$  and choose  $n$  large enough such that  $||x_i|| \leq |\pi|^{-n}$  for all  $i$ . By (3.2) the symbols of the elements  $\pi^n x_i$  in  $gr_{\bullet}^F A_n$  are in degree one and constitute a complete set of variables over  $F_0 A_n$ . Repeating the argument above for  $A_n$  shows that  $||\pi^n x_i||_n = 1$  for all  $i$  and that

$$||a||_n = \max_{\underline{k}} |a_{\underline{k}}| \cdot \prod_i ||x_i||_n^{k_i} = \max_{\underline{k}} |a_{\underline{k}}| \cdot \prod_i |\pi|^{-nk_i}.$$

Our assertion follows now from

$$||a|| \leq \max_{\underline{k}} |a_{\underline{k}}| \cdot ||\underline{x}^{\underline{k}}|| \leq \max_{\underline{k}} |a_{\underline{k}}| \prod_i ||x_i||^{k_i} \leq ||a||_n.$$

□

**Proposition 3.6.** *The canonical homomorphism  $A_K \rightarrow \hat{A}_K$  is faithfully flat.*

Before we turn to the proof of the proposition we establish two auxiliary lemmas. We consider the  $\pi$ -adic filtration on  $A_n, \hat{A}_n$  and  $\hat{A}_{n,K}$ . Let  $gr_{\bullet}^{\pi} A_n$  be the associated graded ring of  $A_n$  and let  $t$  be the principal symbol of  $\pi$ . Of course,  $gr_{\bullet}^{\pi} A_n = gr_{\bullet}^{\pi} \hat{A}_n$ . As we have explained above  $gr_{\bullet}^{\pi} A_n = (A_n/\pi A_n)[t]$  equals the polynomial ring over  $A_n/\pi A_n$  in the variable  $t$ . In particular,

$$gr_{\bullet}^{\pi} \hat{A}_{n,K} = (A_n/\pi A_n)[t^{\pm 1}].$$

Since  $\text{Gr}(\hat{A}_{n,K})$  is noetherian, the ring  $A_n/\pi A_n$  is noetherian, too. So  $gr_{\bullet}^{\pi} A_n$  is noetherian. Since  $A_n$  is  $R$ -flat, the  $\pi$ -adic filtration on  $A_n$  is separated. Since  $\pi$  is a central and regular element in  $A_n$ , we have the Artin-Rees property for the  $\pi$ -adic filtration on  $A_n$  [21, Cor.



I.4.4.8]. This implies that the Rees ring associated with the  $\pi$ -adic filtration of  $A_n$  is noetherian [21, Thm. II.1.1.5] and this finally allows us to apply the theory of lifted Ore sets as explained in [22]. To do this, let  $T_n \subseteq gr_{\bullet}^{\pi} A_n$  be the central and multiplicative subset equal to  $\{1, t, t^2, \dots\}$  and put

$$S_n := \{s \in A_n : \sigma(s) \in T\}.$$

Here,  $\sigma$  denotes the principal symbol map for the  $\pi$ -adic filtration on  $A_n$ . One has  $S_n = \{\pi^m(1 + I_n) : m \geq 0\}$  where  $I_n$  denotes the ideal of  $A_n$  generated by  $\pi$ . Recall the notion of an Ore set in a (noncommutative) ring [24, 2.1.13].

**Lemma 3.7.** *The set  $S_n$  is an Ore set in  $A_n$ . There is a filtration on the localization  $S_n^{-1}A_n$  making  $A_n \rightarrow S_n^{-1}A_n$  a filtered homomorphism. The associated graded ring is canonically isomorphic to the localization  $T_n^{-1}(gr_{\bullet}^{\pi} A_n)$ . The completion homomorphism  $S_n^{-1}A_n \rightarrow \widehat{S_n^{-1}A_n}$  is faithfully flat.*

*Proof.* The statements about the Ore set, the filtration and the graded ring follow from [22, Cor. 2.2/Cor. 2.4]. Note that the filtration on  $S_n^{-1}A_n$  is Zariskian in the sense of [21] and therefore  $S_n^{-1}A_n \rightarrow \widehat{S_n^{-1}A_n}$  is indeed faithfully flat [21, Thm. II.2.1.2].  $\square$

**Lemma 3.8.** *In the situation of the preceding lemma, the canonical homomorphism  $A_n \rightarrow \hat{A}_{n,K}$  extends to an isomorphism of  $K$ -algebras*

$$\widehat{S_n^{-1}A_n} \xrightarrow{\simeq} \hat{A}_{n,K}.$$

*Proof.* The canonical homomorphism  $h : A_n \rightarrow \hat{A}_{n,K}$  is of course filtered relative to  $\pi$ -adic filtrations. Moreover,  $h(1 + I_n)$  consists of units in  $\hat{A}_n$  which implies  $h(s) \in (\hat{A}_{n,K})^{\times}$  for each  $s \in S_n$ . For any  $m$  we denote the homogeneous component of  $gr_{\bullet}^{\pi} A_n$  of degree  $m$  by  $gr_m^{\pi} A_n$ , and similarly for the graded rings  $gr_{\bullet}^{\pi} \hat{A}$  and  $gr_{\bullet}^{\pi} \hat{A}_{n,K}$ . Given  $s \in S_n$  with  $\sigma(s) \in gr_m^{\pi} A_n$  we have  $\sigma(h(s)) \in gr_m^{\pi} \hat{A}_{n,K}$ . We have already explained that  $A_n/\pi A_n$  is an integral domain. Hence, the graded ring  $gr_{\bullet}^{\pi} \hat{A}_{n,K} = (A_n/\pi A_n)[t^{\pm 1}]$  is an integral domain, too, and therefore its principal symbol map is multiplicative. Since  $\sigma(1) = 1 \in gr_0^{\pi} \hat{A}_{n,K}$ , we deduce that  $\sigma(h(s)^{-1}) \in gr_{-m}^{\pi} \hat{A}_{n,K}$ . The universal property of microlocalization [21, Prop. IV.1.1.3] applied to  $h$  therefore yields a filtered homomorphism

$$\hat{h} : \widehat{S_n^{-1}A_n} \rightarrow \hat{A}_{n,K}$$

such that  $h = \hat{h} \circ q$  where  $q$  equals the canonical map  $A_n \rightarrow \widehat{S_n^{-1}A_n}$ . We claim that  $\hat{h}$  is an isomorphism. Since the filtrations on source and target are exhaustive, separated and complete, it suffices to check that its graded map is an isomorphism [21, Cor. I.4.2.5]. However this graded map equals the canonical map between the graded ring of  $S_n^{-1}A_n$  and  $gr_{\bullet}^{\pi} \hat{A}_{n,K} = T_n^{-1}(gr_{\bullet}^{\pi} A_n)$  which is an isomorphism according to the preceding lemma.  $\square$

We now turn to the proof of the proposition.

*Proof.* Consider a (left) ideal  $J \subset A_K$ . Since  $A_K = A_n \otimes_{o_K} K \rightarrow \hat{A}_{n,K}$  is flat, the map

$$(3.9) \quad \hat{A}_{n,K} \otimes_{A_K} J \longrightarrow \hat{A}_{n,K}$$

is injective for any  $n$ . The ring  $A_K$  being noetherian, the  $A_K$ -module  $J$  is finitely presented and, hence, so is the  $\hat{A}_K$ -module  $\hat{A}_K \otimes_{A_K} J$ . It is therefore a coadmissible module for the Fréchet-Stein algebra  $\hat{A}_K$  [35, Cor. 3.4] and, consequently, equals the projective limit over the modules  $\hat{A}_{n,K} \otimes_{A_K} J$ . Since the projective limit is left-exact, we obtain thereby from (3.9) the injectivity of the map  $\hat{A}_K \otimes_{A_K} J \rightarrow \hat{A}_K$ . This establishes the flatness of the map  $A_K \rightarrow \hat{A}_K$ .

We turn to faithful flatness. To this end, consider a (left)  $A_K$ -module  $M$  and assume  $\hat{A}_K \otimes_{A_K} M = 0$ . Since  $A_K \rightarrow \hat{A}_K$  is flat, we may assume [5, 3.3.5] that  $M$  is a cyclic module on one generator, say  $m$ . According to the first lemma, the completion homomorphism  $S_n^{-1}A_n \rightarrow \widehat{S_n^{-1}A_n}$  is faithfully flat. Moreover,  $\pi \in S_n$ , so that  $S_n^{-1}A_n = S_n^{-1}A_K$ . According to the second lemma, we have an isomorphism  $\widehat{S_n^{-1}A_n} \simeq \hat{A}_{n,K}$ . We may therefore deduce from

$$\widehat{S_n^{-1}A_n} \otimes_{S_n^{-1}A_n} S_n^{-1}M = \hat{A}_{n,K} \otimes_{A_K} M = \hat{A}_{n,K} \otimes_{\hat{A}_K} (\hat{A}_K \otimes_{A_K} M) = 0$$

that  $S_n^{-1}M = 0$ , in other words,  $M$  is  $S_n$ -torsion for all  $n$ . Thus, there exists an element  $f_n \in S_n$  with  $f_n m = 0$  for all  $n$ . However,  $S_n$  is of the form  $\cup_{m \geq 0} \pi^m \cdot (1 + I_n)$  where  $I_n$  denotes the ideal generated by  $\pi$  in  $A_n = \sum_{i \geq 0} \pi^{ni} F_i A$ . Since  $\hat{A}_{n,K}$  is  $\pi$ -adically complete, the elements in  $1 + \pi F_0 A$  are units in  $\hat{A}_{n,K}$  which allows us to assume that  $f_n$  is of the form  $1 + \pi^n g_n$  with some element  $g_n \in A$ . Since  $Am$  is contained in the  $K$ -vector space  $M$ , it is  $R$ -free and hence  $\pi$ -adically separated. The limit of the sequence  $f_n m \in Am$  in the  $\pi$ -adic topology equals  $m$ . Thus,  $m = 0$  and  $M = 0$ . This completes the proof of the proposition.  $\square$

**Corollary 3.10.** *Let  $M$  be a finitely generated  $A_K$ -module and  $\hat{M} := \hat{A}_K \otimes_{A_K} M$ . Then*

$$j_{A_K}(M) = j_{\hat{A}_K}(\hat{M}) .$$

*Proof.* Use Prop. 3.6 and Lem. 2.2.  $\square$

We have already explained that the ring  $\text{Gr}(\hat{A}_{n,K})$  is a noetherian regular integral domain. Let  $d$  denote its (finite) global dimension. Of course,  $d$  equals the sum of the global dimension of  $F_0 A / \pi F_0 A$  and the number  $r$  as defined in (3).

**Proposition 3.11.** *The noetherian ring  $\hat{A}_{n,K}$  is Auslander regular of global dimension  $\leq d$ .*

*Proof.* The ring  $\text{Gr}(\hat{A}_{n,K})$  is Auslander regular [21, III.2.4.3] and therefore  $A_n / \pi A_n$  is Auslander regular of global dimension  $\leq d$  according to [21, II.3.1.4] and [21, III.2.2.5].

According to [21, III.3.4.6] we obtain that the rings  $gr_{\bullet}^{\pi} \hat{A}_n = (A_n/\pi A_n)[t]$  and  $gr_{\bullet}^{\pi} \hat{A}_{n,K} = (A_n/\pi A_n)[t^{\pm 1}]$  are Auslander regular of global dimension  $\leq d + 1$ . A second application of [21, II.3.1.4] and [21, III.2.2.5] now yields that the rings  $\hat{A}_n$  and  $\hat{A}_{n,K}$  are Auslander regular of global dimension  $\leq d + 1$ . On the other hand,  $\pi \hat{A}_n$  is contained in the Jacobson radical of  $\hat{A}_n$  according to [21, I.3.3.5] and so  $\pi$  annihilates any simple  $\hat{A}_n$ -module. Hence the global dimension of  $\hat{A}_{n,K} = \hat{A}_n[\pi^{-1}]$  is even  $\leq d$  by [24, 7.4.3/7.4.4].  $\square$

According to the proposition the Fréchet-Stein algebra  $\hat{A}_K$  verifies that assumption (DIM) as formulated in [35, 8.8]. Consequently, the grade number  $j_{\hat{A}_K}$  is a well-behaved codimension function on the abelian category of coadmissible modules. This implies the following corollary, cf. [35, Lem. 8.4].

**Corollary 3.12.** *If  $M$  is a coadmissible  $\hat{A}_K$ -module and  $M_n := \hat{A}_{n,K} \otimes_{\hat{A}_K} M$ , then*

$$j_{\hat{A}_K}(M) = \min_n j_{\hat{A}_{n,K}}(M_n) .$$

We finish with a discussion of examples of algebras  $A$  satisfying our requirements. Let  $\mathfrak{g}$  be a  $R$ -Lie algebra which is finite and free as an  $R$ -module, say of rank  $d$ . Let  $A := U(\mathfrak{g})$  be its universal enveloping algebra equipped with its usual positive filtration. Then  $A$  satisfies all our requirements. Indeed, (1) follows by definition of the filtration and (2) is trivial since  $F_0 A = R$ . It is well-known that the graded ring of  $U(\mathfrak{g})$  equals the symmetric algebra of the  $R$ -module  $\mathfrak{g}$  whence (3). Note that  $A_K = U(\mathfrak{g}_K)$  with the  $K$ -Lie algebra  $\mathfrak{g}_K := \mathfrak{g} \otimes_R K$  and that  $\hat{A}_{n,K} = \hat{U}(\pi^n \mathfrak{g})_K$ , i.e.  $\hat{A}_{n,K}$  coincides with the  $\pi$ -adic completion with subsequent inversion of  $\pi$  of the universal enveloping algebra  $U(\pi^n \mathfrak{g})$  of the  $R$ -Lie algebra  $\pi^n \mathfrak{g}$  for all  $n$ . Note that  $\text{Gr}(\hat{A}_{n,K})$  is isomorphic to the symmetric algebra of the  $R/\pi R$ -vector space  $\mathfrak{g}/\pi \mathfrak{g}$ . In particular, the global dimension of  $\hat{U}(\pi^n \mathfrak{g})_K$  is in fact equal to  $d$  as follows from [2, Prop. 9.1] applied to the augmentation character  $\hat{U}(\pi^n \mathfrak{g})_K \rightarrow K$  given by  $x = 0$  for all  $x \in \pi^n \mathfrak{g}$ . Since  $F_0 A = R$ , the Arens-Michael envelope  $\hat{A}_K$  equals the completion of  $U(\mathfrak{g}_K)$  with respect to *all* submultiplicative seminorms on the abstract  $K$ -algebra  $U(\mathfrak{g}_K)$ . This completion was first introduced and studied in [29] and [30]. For future reference we restate its faithful flatness property.

**Theorem 3.13.** *The natural homomorphism  $U(\mathfrak{g}_K) \rightarrow \hat{U}(\mathfrak{g}_K)$  is faithfully flat.*

As a second example we consider a smooth affine integral scheme  $X$  of finite type over  $R$  whose closed fibre is integral. We assume that the locally free module of differentials  $\Omega_{X/R}$  is already free, say of rank  $d$ . Let  $A := \mathcal{D}(X)$  be the ring of (crystalline) global differential operators on  $X$  with its natural filtration.<sup>2</sup> In particular,  $F_0 A = \mathcal{O}(X)$ , the ring of global sections of  $X$ . Then  $A$  satisfies all our requirements: again, (1) follows by definition of the filtration and (2) follows from  $F_0 A = \mathcal{O}(X)$  and our assumptions on

<sup>2</sup>Note that  $\mathcal{D}(X)$  coincides with the *derivation ring* of  $\mathcal{O}(X)$  as studied in [24, 15.1].

$X$ . It is well-known that the graded ring of  $\mathcal{D}(X)$  equals the symmetric algebra of the  $\mathcal{O}(X)$ -module consisting of the global vector fields on  $X$  whence (3).

More generally, the enveloping algebra of a Lie algebroid [27] gives rise to many examples. Let us briefly recall the definition (taken from [1]). Let  $R \rightarrow S$  be a ring homomorphism to some commutative ring  $S$ . A *Lie algebroid* is a pair  $(L, a)$  consisting of an  $R$ -Lie algebra and  $S$ -module  $L$ , together with an  $S$ -linear  $R$ -Lie algebra homomorphism  $a$  from  $L$  to the  $R$ -linear derivations of  $S$ , such that  $[v, sw] = s[v, w] + a(v)(s)w$  for all  $v, w \in L$  and  $s \in S$ . It is possible to form a unital associative  $R$ -algebra  $U(L)$  called the *enveloping algebra* of  $(L, a)$  which is generated as a  $R$ -algebra by  $S$  and  $L$  subject to appropriate natural relations. Whenever  $L$  is a projective  $S$ -module,  $U(L)$  has a natural positive filtration with associated graded ring the symmetric algebra  $\text{Sym}_S(L)$ . Suppose now that  $L$  is already a free  $S$ -module, say of rank  $d$ . Then  $F_0 A = S$  and  $A := U(L)$  satisfies all our requirements if and only if  $F_0 A = S$  satisfies (2). Our two first examples above are the special cases  $S := R$  and  $(L, a) := (\mathfrak{g}, 0)$  respectively  $S := \mathcal{O}(X)$  and  $(L, a) := (\Omega_{X/R}^\vee(X), id)$ .

#### 4. FROM $D(\mathfrak{g}, P)$ -MODULES TO $D(G)$ -MODULES

We consider the locally  $L$ -analytic groups  $P$  and  $G$  as well as the maximal compact subgroup  $G_0 \subseteq G$ . We let  $P_0 = G_0 \cap P$ . The locally analytic distribution algebras with coefficients in  $K$  are denoted by  $D(P), D(G), D(P_0)$  and  $D(G_0)$ . In this section, we will consider a certain functor  $\mathcal{F}_P^G(\cdot)'$  from Lie algebra representations of  $\mathfrak{g}$  endowed with a compatible locally analytic action of  $P$  to locally analytic  $G$ -representations. This functor, or rather its restriction to certain highest weight categories was introduced and studied in [26]. To alleviate notation, we denote the universal enveloping algebra of the base change to  $K$  of the  $L$ -Lie algebra  $\mathfrak{g}$  by  $U(\mathfrak{g})$ .

The group  $G$  and its subgroup  $P$  act via the adjoint representation on the Lie algebra  $\mathfrak{g}$ . We denote by

$$D(\mathfrak{g}, P) := D(P) \otimes_{U(\mathfrak{p})} U(\mathfrak{g})$$

the corresponding skew-product ring. Similarly, we denote by  $D(\mathfrak{g}, P_0)$  the skew-product ring  $D(P_0) \otimes_{U(\mathfrak{p})} U(\mathfrak{g})$ .

**Lemma 4.1.** *The natural linear map  $D(\mathfrak{g}, P) \rightarrow D(G)$  is an injective ring homomorphism with image equal to the subring of  $D(G)$  generated by  $D(P)$  and  $U(\mathfrak{g})$ .*

*Proof.* Let  $U_P^-$  be the group of points of the opposite unipotent radical of  $P$  and let  $\mathfrak{u}_P^-$  be its Lie-algebra. In particular,  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}_P^-$ . The multiplication map  $P \times U_P^- \rightarrow G$  is injective and induces an injective homomorphism  $D(P \times U_P^-) \rightarrow D(G)$ . The linear map appearing in the lemma is injective being the composite of the injective linear maps

$$D(\mathfrak{g}, P) = D(P) \otimes_K U(\mathfrak{u}_P^-) \longrightarrow D(P) \otimes_K D(U_P^-) \longrightarrow D(P \times U_P^-) \longrightarrow D(G) .$$

The remaining assertions are clear.  $\square$

An obvious variant of the above proof for the group  $G_0$  shows that the natural linear map  $D(\mathfrak{g}, P_0) \rightarrow D(G_0)$  is an injective ring homomorphism with image equal to the subring of  $D(G_0)$  generated by  $D(P_0)$  and  $U(\mathfrak{g})$ .

**Lemma 4.2.** *One has*

$$D(G) = D(G_0) \otimes_{D(\mathfrak{g}, P_0)} D(\mathfrak{g}, P)$$

as bimodules. In particular,

$$D(G) \otimes_{D(\mathfrak{g}, P)} M = D(G_0) \otimes_{D(\mathfrak{g}, P_0)} M$$

for any  $D(\mathfrak{g}, P)$ -module  $M$ .

*Proof.* The bimodule map equal to the composite

$$D(G_0) \otimes_{D(P_0)} D(P) \longrightarrow D(G_0) \otimes_{D(\mathfrak{g}, P_0)} D(\mathfrak{g}, P) \rightarrow D(G)$$

is an isomorphism according to [36, Lem. 6.1]. Since the first map is surjective, both individual maps are isomorphisms as well. The second statement is clear.  $\square$

We consider the functor

$$M \mapsto \mathcal{F}_P^G(M)' := D(G) \otimes_{D(\mathfrak{g}, P)} M$$

from  $D(\mathfrak{g}, P)$ -modules to  $D(G)$ -modules. Here, we follow the notation of [26], compare in particular Prop. 3.7 in loc.cit. If the parabolic subgroup  $P$  is clear from the context, we will occasionally abbreviate

$$\mathbf{M} := \mathcal{F}_P^G(M)' .$$

**Lemma 4.3.** *If  $M$  is finitely generated as  $U(\mathfrak{g})$ -module, then  $\mathbf{M}$  is coadmissible.*

*Proof.* As a  $D(G_0)$ -module we have  $\mathbf{M} = D(G_0) \otimes_{D(\mathfrak{g}, P_0)} M$  according to the preceding lemma. The group  $P_0$  is topologically finitely generated. Let  $p_1, \dots, p_r$  be a set of topological generators and let  $m_1, \dots, m_s$  be a set of generators for the  $U(\mathfrak{g})$ -module  $M$ . Since  $U(\mathfrak{g})$  is noetherian, the  $D(G_0)$ -module  $D(G_0) \otimes_{U(\mathfrak{g})} M$  is finitely presented and hence coadmissible. Consider its submodule  $N$  finitely generated by the elements  $\delta_{p_i} \otimes m_j - 1 \otimes \delta_{p_i} m_j$ . Then  $N$  is coadmissible and it suffices to see that  $N$  equals the kernel of the natural surjection  $D(G_0) \otimes_{U(\mathfrak{g})} M \rightarrow \mathbf{M}$ . To this end, observe that  $M$  equals a countable union of vector subspaces of finite dimension. We give  $M$  the finest locally convex topology and let  $W := D(G_0) \otimes_K M$  have its projective tensor product topology, e.g. [32, §17B]. Then  $W$  satisfies the assumptions of [32, Prop. 8.8] and so the surjective continuous linear map  $W \rightarrow D(G_0) \otimes_{U(\mathfrak{g})} M$  is open. In other words, the canonical topology on the coadmissible

module  $D(G_0) \otimes_{U(\mathfrak{g})} M$  equals the quotient topology of  $W$  by a suitable closed subspace. Hence, if  $\delta_n \rightarrow \delta$  is a convergent sequence in  $D(P_0)$ , then for any  $m \in M$

$$(\delta_n \otimes m - 1 \otimes \delta_n m) \rightarrow (\delta \otimes m - 1 \otimes \delta m)$$

is a convergent sequence in the coadmissible module  $D(G_0) \otimes_{U(\mathfrak{g})} M$ . The abstract group ring  $K[P_0]$  is dense in  $D(P_0)$  according to [34, Lem. 3.1] which implies the claim.  $\square$

We now start a more detailed analysis of the module  $\mathbf{M}$  closely following the discussion in [26, 5.5]. We put

$$\kappa = \begin{cases} 1 & , \quad p > 2 \\ 2 & , \quad p = 2 \end{cases}$$

Let in the following  $r$  always denote a real number in  $(0, 1) \cap p^{\mathbb{Q}}$  with the property:

$$(4.4) \quad \text{there is } m \in \mathbb{Z}_{\geq 0} \text{ such that } s = r^{p^m} \text{ satisfies } \frac{1}{p} < s \text{ and } s^\kappa < p^{-1/(p-1)} .$$

For such numbers  $r$  we let  $D_r(G_0)$  and  $D_r(P_0)$  be the Banach algebras appearing in loc.cit. Let us briefly sketch their construction. One chooses suitable uniform pro- $p$  groups  $H \subset G_0$  and  $H^+ := H \cap P_0$  such that  $H$  is open normal in  $G_0$ . The distribution algebras of  $H$  and  $H^+$  admit canonical  $r$ -norms coming from the canonical  $p$ -valuation on the group [35]. The rings  $D(G_0)$  resp.  $D(P_0)$  are finite free ring extensions over  $D(H)$  resp.  $D(H^+)$  and carry the corresponding maximum norms. The rings  $D_r(G_0)$  resp.  $D_r(P_0)$  are the associated completions. They define the Fréchet-Stein structure of  $D(G_0)$  resp.  $D(P_0)$ . Let  $U_r(\mathfrak{g})$  and  $U_r(\mathfrak{p})$  be the topological closure of  $U(\mathfrak{g})$  in  $D_r(G_0)$  and  $U(\mathfrak{p})$  in  $D_r(P_0)$  respectively. Put

$$D_r(\mathfrak{g}, P_0) := D_r(P_0) \otimes_{U_r(\mathfrak{p})} U_r(\mathfrak{g}) .$$

An argument completely analogous to Lem. 4.1 shows that the natural linear map  $D_r(\mathfrak{g}, P_0) \rightarrow D_r(G_0)$  is an injective ring homomorphism with image equal to the subring of  $D_r(G_0)$  generated by  $D_r(P_0)$  and  $U_r(\mathfrak{g})$ . If  $HP_0$  denotes the subgroup of  $G_0$  generated by  $H$  and  $P_0$ , the intersection

$$P_{0,r} := HP_0 \cap D_r(\mathfrak{g}, P_0)$$

is thus well-defined.

**Lemma 4.5.** *The set  $P_{0,r}$  is an open normal subgroup of  $HP_0$ . One has*

$$D_r(G_0) = \bigoplus_{g \in G_0/P_{0,r}} \delta_g D_r(\mathfrak{g}, P_0) .$$

*Proof.* This follows from [26, 5.6].  $\square$

We let

$$M_r := U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M, \quad \mathbf{M}_r := D_r(G_0) \otimes_{D(G_0)} \mathbf{M} = D_r(G_0) \otimes_{D(\mathfrak{g}, P_0)} M.$$

For  $g \in G$  we denote by  $\text{Ad}(g)$  the automorphism of  $U(\mathfrak{g})$  (or  $U_r(\mathfrak{g})$ ) induced by the left conjugation action  $h \mapsto ghg^{-1}$  of  $g$  on  $G$ . We note that the group  $P_0$  acts on  $M_r$  via  $p \cdot (\mathfrak{x} \otimes m) := (\text{Ad}(p)(\mathfrak{x})) \otimes pm$ .

**Lemma 4.6.** *The natural map*

$$M_r \xrightarrow{\simeq} D_r(\mathfrak{g}, P_0) \otimes_{D(\mathfrak{g}, P_0)} M$$

*induced from the map  $U_r(\mathfrak{g}) \rightarrow D_r(\mathfrak{g}, P_0), \mathfrak{x} \mapsto 1 \otimes \mathfrak{x}$  is bijective.*

*Proof.* The ring  $D_r(P_0)$  is a finite and free module over  $U_r(\mathfrak{p})$  on a basis given by distributions  $\delta_p$  with  $p \in P_0$ . The  $(P_0, U_r(\mathfrak{g}))$ -module structure on  $M_r$  therefore extends to a module structure over the ring  $D_r(\mathfrak{g}, P_0)$ . The resulting map  $D_r(\mathfrak{g}, P_0) \otimes_{D(\mathfrak{g}, P_0)} M \rightarrow M_r$  provides an inverse for the map in question.  $\square$

Using the two lemmas we can derive the following decomposition of  $\mathbf{M}_r$  as  $U_r(\mathfrak{g})$ -module,

$$(4.7) \quad \mathbf{M}_r = D_r(G_0) \otimes_{D(\mathfrak{g}, P_0)} M = D_r(G_0) \otimes_{D_r(\mathfrak{g}, P_0)} M_r = \bigoplus_{g \in G_0/P_0, r} \delta_g \star M_r,$$

where  $\delta_g \star M_r$  denotes the twist of the  $U_r(\mathfrak{g})$ -module  $M_r$  with the automorphism  $\text{Ad}(g)$  in the sense of Lem. 2.1.

**Proposition 4.8.** *On the abelian category of  $D(\mathfrak{g}, P)$ -modules which are finitely generated as  $U(\mathfrak{g})$ -modules, the correspondence  $M \mapsto \mathbf{M}$  constitutes an exact and faithful functor with trivial kernel (i.e.  $\mathbf{M} = 0$  implies  $M = 0$ ).*

*Proof.* The projective limit  $\hat{U}(\mathfrak{g}) = \varprojlim_r U_r(\mathfrak{g})$  is Fréchet-Stein and equals the Arens-Michael envelope of  $U(\mathfrak{g})$ . If  $M \neq 0$ , then

$$\hat{M} := \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M \neq 0$$

according to Thm. 3.13. Moreover, since  $M$  is finitely generated, the  $\hat{U}(\mathfrak{g}_K)$ -module  $\hat{M}$  is finitely presented and hence coadmissible. Thus,  $\hat{M} = \varprojlim_r M_r$  which implies  $M_r \neq 0$  for a cofinal family of values of  $r$ . According to (4.7),  $\mathbf{M}_r \neq 0$  for these  $r$ . Since  $\mathbf{M}$  is coadmissible (4.3), this implies  $\mathbf{M} \neq 0$ . Moreover, a sequence of coadmissible  $D(G_0)$ -modules is exact if and only if this is true after base extension to  $D_r(G_0)$  for a cofinal family of values of  $r$ . The decomposition (4.7) is natural in  $M$ . Since the functor  $M \mapsto M_r$  is exact [35, Rem. 3.2], so is the functor  $M \mapsto \mathbf{M}$ . The faithfulness is now a formal consequence.  $\square$



We compute a class of examples related to *locally analytic parabolic induction*. Recall the Levi decompositions  $P = L_P \cdot U_P$  and  $\mathfrak{p} = \mathfrak{l}_P \oplus \mathfrak{u}_P$ . Let  $V$  be a locally analytic  $L_P$ -representation on a finite-dimensional  $K$ -vector space. We set  $\mathfrak{u}_P V = 0$  and consider  $V$  a  $U(\mathfrak{p})$ -module. The induced  $U(\mathfrak{g})$ -module  $M(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$  is then naturally a  $D(\mathfrak{g}, P)$ -module which is finitely generated over  $U(\mathfrak{g})$ . Indeed, we have the diagonal action of  $L_P$  on the tensor product  $M(V)$  where  $L_P$  acts on the factor  $U(\mathfrak{g})$  via the adjoint action. It extends to a  $D(L_P)$ -action and it suffices therefore to check that the  $\mathfrak{u}_P$ -action extends compatibly to  $D(U_P)$ . However, the action of the Lie algebra  $\mathfrak{u}_P$  even integrates uniquely to an algebraic action of  $U_P$  on  $M(V)$  as follows. Given an element  $u = \exp(\mathfrak{x}) \in \mathbf{U}_P(\overline{K})$ , where  $\overline{K}$  denotes an algebraic closure of  $K$ , we define  $\rho(u) := \sum_{n \geq 0} \frac{\rho(\mathfrak{x})^n}{n!}$ , where  $\rho(\mathfrak{x})^n = 0$  for  $n \gg 0$ . The representations of  $L_P$  and  $U_P$  are compatible in the sense that  $h \circ \rho(u) \circ h^{-1} = \rho(\text{Ad}(h)(u))$ , for  $h \in L_P$ ,  $u \in U_P$ . Hence,  $M(V)$  is a  $D(P)$ -module and then even a  $D(\mathfrak{g}, P)$ -module as claimed.

**Proposition 4.9.** *The map  $V \rightarrow M(V), v \mapsto 1 \otimes v$  induces an isomorphism of  $D(G)$ -modules*

$$D(G) \otimes_{D(P)} V \xrightarrow{\cong} \mathcal{F}_P^G(M(V))' .$$

*Proof.* The map

$$\mathcal{F}_P^G(M(V))' \rightarrow D(G) \otimes_{D(P)} V, \delta \otimes (x \otimes v) \mapsto (\delta x) \otimes v$$

for  $\delta \in D(G), x \in U(\mathfrak{g}), v \in V$  is well-defined and provides a two-sided inverse.  $\square$

Remark: The module  $D(G) \otimes_{D(P)} V$  is dual to the locally analytic parabolic induction  $\text{Ind}_P^G(V')$ .

In the following we will investigate the behavior of the functor  $\mathcal{F}_P^G(\cdot)'$  in terms of dimensions. To this end, recall that the ring  $U(\mathfrak{g})$  is a noetherian Auslander regular ring of global dimension  $d := \dim_L \mathfrak{g}$ . For a finitely generated  $U(\mathfrak{g})$ -module  $M$  we therefore have its canonical dimension  $\dim_{U(\mathfrak{g})} M := d - j_{U(\mathfrak{g})}(M)$ , cf. 2.

Remark: Traditionally, dimension theory over the ring  $U(\mathfrak{g})$  is developed using the so-called *Gelfand-Kirillov dimension*, cf. [16]. However, it follows from [20, Remark 5.8 (3)] together with [24, Prop. 8.1.15 (iii)] that for finitely generated  $U(\mathfrak{g})$ -modules, Gelfand-Kirillov dimension coincides with canonical dimension.

On the other hand, for any compact open subgroup  $H \subseteq G$  and a coadmissible  $D(H)$ -module  $M$ , we define

$$\dim_{D(H)} M := d - j_{D(H)}(M) .$$

If  $D(H) = \varprojlim_r D_r(H)$  is a Fréchet-Stein structure for  $D(H)$  and  $M_r := D_r(H) \otimes_{D(H)} M$ , then

$$(4.10) \quad \dim_{D(H)}(M) = \sup_r \dim_{D_r(H)}(M_r)$$

according to [35, §8]. Moreover, if  $M$  is even a  $D(G)$ -module, then, according to [35] and [28], the number  $\dim_{D(H)} M$  is independent of the choice of  $H$ . In this case, we denote it by  $\dim_{D(G)} M$ , or simply  $\dim M$ , if no confusion can arise, and call it the *canonical dimension* of the coadmissible  $D(G)$ -module  $M$ .

We shall also need the Arens-Michael envelope  $\hat{U}(\mathfrak{g})$  of  $\mathfrak{g}$  as introduced in the preceding section. Recall that this is a Fréchet-Stein algebra equal to the completion of  $U(\mathfrak{g})$  with respect to all submultiplicative seminorms on  $U(\mathfrak{g})$ . As such, it comes with a natural completion homomorphism  $U(\mathfrak{g}) \rightarrow \hat{U}(\mathfrak{g})$  which is faithfully flat, cf. Thm. 3.13.

**Theorem 4.11.** *If  $M$  is a  $D(\mathfrak{g}, P)$ -module which is finitely generated as  $U(\mathfrak{g})$ -module, then*

$$\dim_{D(G)} \mathbf{M} = \dim_{U(\mathfrak{g})} M .$$

*Proof.* It suffices to prove  $j_{D(G_0)}(\mathbf{M}) = j_{U(\mathfrak{g})}(M)$ . The left-hand side of this identity equals  $\min_r j_{D_r(G_0)}(\mathbf{M}_r)$  according to (4.10). Now  $D_r(G_0)$  is a finite free  $U_r(\mathfrak{g})$ -module on a basis which consists of units satisfying the assumptions of [35, Lem. 8.8]. Hence,  $j_{D_r(G_0)}(\mathbf{M}_r) = j_{U_r(\mathfrak{g})}(\mathbf{M}_r)$  for all  $r$ . By (4.7) together with Lem. 2.1, we have

$$j_{U_r(\mathfrak{g})}(\mathbf{M}_r) = \max_{g \in G_0/P_{0,r}} j_{U_r(\mathfrak{g})}(\delta_g \star M_r) = j_{U_r(\mathfrak{g})} M_r .$$

So it remains to show  $j_{U(\mathfrak{g})}(M) = \min_r j_{U_r(\mathfrak{g})}(M_r)$ . Since  $\hat{M} := \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$  is coadmissible, we have

$$j_{U(\mathfrak{g})}(M) = j_{\hat{U}(\mathfrak{g})}(\hat{M}) = \min_r j_{U_r(\mathfrak{g})}(M_r)$$

according to (3.10) and (3.12). □

Combining the theorem with [16, Lem. 8.9] gives the dimension of parabolically induced representations.

**Corollary 4.12.** *One has  $\dim \mathcal{F}_P^G(M(V))' = \dim_L(\mathfrak{g}/\mathfrak{p})$  where  $\dim_L$  denotes vector space dimension.*

## 5. HIGHEST WEIGHT MODULES AND DIMENSION

In this section we explain the relation to the parabolic BGG-categories for the pair  $\mathfrak{p} \subseteq \mathfrak{g}$  appearing in [26] and compute the dimensions of certain irreducible  $G$ -representations occurring in the image of the functor  $\mathcal{F}_P^G$ . As in the previous section, we make the general

convention that, when dealing with universal enveloping algebras, we write  $U(\mathfrak{g})$ ,  $U(\mathfrak{p})$  etc. to denote the corresponding universal enveloping algebras *after base change to  $K$* , i.e., what is precisely  $U(\mathfrak{g}_K)$ ,  $U(\mathfrak{p}_K)$  and so on.

**5.1. The category  $\mathcal{O}$  and its parabolic variants  $\mathcal{O}^p$ .** The category  $\mathcal{O}$  in the sense of Bernstein, Gelfand, Gelfand, cf. [4], [15], is defined for complex semi-simple Lie algebras. Here we consider the following variant for split reductive Lie algebras over a field of characteristic zero. Thus we let  $\mathcal{O}$  be the full subcategory of all  $U(\mathfrak{g})$ -modules  $M$  which satisfy the following properties:

- (1)  $M$  is finitely generated as a  $U(\mathfrak{g})$ -module.
- (2)  $M$  decomposes as a direct sum of one-dimensional  $\mathfrak{t}_K$ -representations.
- (3) The action of  $\mathfrak{b}_K$  on  $M$  is locally finite, i.e. for every  $m \in M$ , the subspace  $U(\mathfrak{b}) \cdot m \subset M$  is finite-dimensional over  $K$ .

As in the classical case one shows that  $\mathcal{O}$  is a  $K$ -linear, abelian, noetherian, artinian category which is closed under submodules and quotients, cf. [15, 1.1, 1.11]. In particular, every object of  $\mathcal{O}$  has a Jordan-Hölder series and a simple object of  $\mathcal{O}$  is simple as abstract  $U(\mathfrak{g})$ -module.

Following [26] we define a certain 'algebraic' subcategory of  $\mathcal{O}$ . Note that by property (2), we may write any object  $M$  in  $\mathcal{O}$  as a direct sum

$$(5.1) \quad M = \bigoplus_{\lambda \in \mathfrak{t}_K^*} M_\lambda$$

where  $M_\lambda = \{m \in M \mid \forall \mathfrak{x} \in \mathfrak{t}_K : \mathfrak{x} \cdot m = \lambda(\mathfrak{x})m\}$  is the  $\lambda$ -eigenspace attached to  $\lambda \in \mathfrak{t}_K^* = \text{Hom}_K(\mathfrak{t}_K, K)$ . Let  $X^*(\mathbf{T}) = \text{Hom}(\mathbf{T}, \mathbb{G}_m)$  be the group of characters of the torus  $\mathbf{T}$  which we consider via the derivative as a subgroup of  $\mathfrak{t}_K^*$ .

We denote by  $\mathcal{O}_{\text{alg}}$  the full subcategory of  $\mathcal{O}$  whose consisting of objects  $M \in \mathcal{O}$  where the  $\mathfrak{t}_K$ -module structure on every  $M_\lambda$  lifts to an algebraic action of  $\mathbf{T}$ . Again,  $\mathcal{O}_{\text{alg}}$  is an abelian noetherian, artinian category which is closed under submodules and quotients. The Jordan-Hölder series of a given  $U(\mathfrak{g})$ -module lying in  $\mathcal{O}_{\text{alg}}$  is the same as the one considered in the category  $\mathcal{O}$ .

**Example 5.2.** For  $\lambda \in \mathfrak{t}_K^*$ , let  $K_\lambda = K$  be the 1-dimensional  $\mathfrak{t}_K$ -module where the action is given by  $\lambda$ . Then  $K_\lambda$  extends uniquely to a  $\mathfrak{b}_K$ -module. Let

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} K_\lambda \in \mathcal{O}$$

be the corresponding Verma module. Denote by  $L(\lambda) \in \mathcal{O}$  its simple quotient. Suppose the character  $\lambda$  integrates to a locally analytic character of  $T$ . As we have explained before Prop.4.9, the module  $M(\lambda)$  is then a  $D(\mathfrak{g}, B)$ -module finitely generated over  $U(\mathfrak{g})$  and the same holds true for  $L(\lambda)$ . In this situation,  $M(\lambda)$  resp.  $L(\lambda)$  is an object of  $\mathcal{O}_{\text{alg}}$  if and only if  $\lambda \in X^*(\mathbf{T})$ .

We shall also need the parabolic versions of the above categories. We define  $\mathcal{O}^{\mathfrak{p}}$  to be the category of  $U(\mathfrak{g})$ -modules  $M$  satisfying the following properties:

- (1)  $M$  is finitely generated as a  $U(\mathfrak{g})$ -module.
- (2) Viewed as a  $\mathfrak{l}_{P,K}$ -module,  $M$  is the direct sum of finite-dimensional simple modules.
- (3) The action of  $\mathfrak{u}_{P,K}$  on  $M$  is locally finite.

This is analogous to the definition over an algebraically closed field, cf. [15, ch. 9]. Clearly, the category  $\mathcal{O}^{\mathfrak{p}}$  is a full subcategory of  $\mathcal{O}$ . Furthermore, it is  $K$ -linear, abelian and closed under submodules and quotients, cf. [15, 9.3]. Hence the Jordan-Hölder series of every  $U(\mathfrak{g})$ -module in  $\mathcal{O}^{\mathfrak{p}} \subset \mathcal{O}$  lies in  $\mathcal{O}^{\mathfrak{p}}$  as well. If  $Q$  is a standard parabolic subgroup with  $Q \supset P$ , then  $\mathcal{O}^{\mathfrak{q}} \subset \mathcal{O}^{\mathfrak{p}}$ . Finally, consider the extreme case  $\mathfrak{p} = \mathfrak{g}$ : the category  $\mathcal{O}^{\mathfrak{g}}$  consists of all finite-dimensional semi-simple  $\mathfrak{g}_K$ -modules. On the other hand,  $\mathcal{O}^{\mathfrak{b}} = \mathcal{O}$ .

Similarly as before we define a subcategory  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  of  $\mathcal{O}^{\mathfrak{p}}$  as follows. Let  $\text{Irr}(\mathfrak{l}_{P,K})^{\text{fd}}$  be the set of isomorphism classes of finite-dimensional irreducible  $\mathfrak{l}_{P,K}$ -modules. Again, any object in  $\mathcal{O}^{\mathfrak{p}}$  has by property (2) a decomposition into  $\mathfrak{l}_{P,K}$ -modules

$$(5.3) \quad M = \bigoplus_{\mathfrak{a} \in \text{Irr}(\mathfrak{l}_{P,K})^{\text{fd}}} M_{\mathfrak{a}}$$

where  $M_{\mathfrak{a}} \subset M$  is the  $\mathfrak{a}$ -isotypic part of the representation  $\mathfrak{a}$ . We denote by  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  the full subcategory of  $\mathcal{O}^{\mathfrak{p}}$  consisting of objects  $M$  of  $\mathcal{O}^{\mathfrak{p}}$  with the following property: if  $M_{\mathfrak{a}} \neq 0$  (with the notation as in 5.3), then  $\mathfrak{a}$  is the Lie algebra representation induced by a finite-dimensional algebraic  $\mathbf{L}_{\mathbf{P},K}$ -representation, where  $\mathbf{L}_{\mathbf{P},K} = \mathbf{L}_{\mathbf{P}} \times_{\text{Spec}(L)} \text{Spec}(K)$ . Again, the category  $\mathcal{O}_{\text{alg}}^{\mathfrak{g}}$  is contained in  $\mathcal{O}_{\text{alg}}$  and contains all finite-dimensional  $\mathfrak{g}_K$ -modules which are induced by algebraic  $\mathbf{G}$ -modules. Every object in  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  has a Jordan-Hölder series which coincides with the Jordan-Hölder series in  $\mathcal{O}_{\text{alg}}$ . If  $M$  is an object of  $\mathcal{O}^{\mathfrak{p}}$ , then  $M$  is in  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  if and only if it is in  $\mathcal{O}_{\text{alg}}$ , cf. [26, Lem. 2.8].

**Example 5.4.** Let  $\Delta$  be the set of simple roots of  $\mathbf{G}$  with respect to  $\mathbf{T} \subset \mathbf{B}$ . Let  $\lambda \in \mathfrak{t}_K^*$  and set  $I = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$ . We let  $\mathbf{P} = \mathbf{P}_I$  be the standard parabolic subgroup of  $\mathbf{G}$  attached to  $I$ . Then  $\lambda$  is dominant with respect to the reductive Lie algebra  $\mathfrak{l}_P$ . Denote by  $V_I(\lambda)$  the corresponding irreducible finite-dimensional  $\mathfrak{l}_{\mathfrak{p}}$ -representation and consider the generalized Verma module (in the sense of Lepowsky [19])

$$M_I(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} V_I(\lambda) .$$

There is a surjective map

$$M(\lambda) \rightarrow M_I(\lambda) ,$$

where the kernel is given by the image of  $\bigoplus_{\alpha \in I} M(s_{\alpha} \cdot \lambda) \rightarrow M(\lambda)$ . Now suppose the  $\mathfrak{l}_{\mathfrak{p}}$ -representation on  $V_I(\lambda)$  integrates to a locally analytic  $L_P$ -representation. As we have

explained before Prop.4.9, the module  $M_I(\lambda)$  is then a  $D(\mathfrak{g}, P)$ -module and finitely generated over  $U(\mathfrak{g})$ . In this situation,  $M_I(\lambda)$  is an object of  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  if and only if the  $\mathfrak{l}_{\mathfrak{p}}$ -action on  $V_I(\lambda)$  integrates to an algebraic  $L_P$ -action. This happens if and only if  $\lambda \in X(\mathbf{T})$ . In this case,  $L(\lambda)$  is an object of  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$ , as well, cf. [15, sec. 9.4].

Let  $M$  be an object of  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  as above. Then  $M$  is the union of finite-dimensional  $\mathfrak{p}_K$ -modules. Denote by  $X$  one of these finite-dimensional submodules. Then  $X$  lifts uniquely to an algebraic  $\mathbf{P}_K$ -representation [26, Cor. 3.6]. Let us sketch the argument. The  $U(\mathfrak{p})$ -module  $X$ , considered as a  $U(\mathfrak{l}_{\mathfrak{p}})$ -module, decomposes into a direct sum of isotypic modules  $X_{\mathfrak{a}}$  and each module  $X_{\mathfrak{a}}$  lifts uniquely to an algebraic representation of  $\mathbf{L}_{\mathbf{P}, K}$ . The action of the Lie algebra  $\mathfrak{u}_{\mathfrak{p}, K}$  integrates uniquely to an algebraic action of  $\mathbf{U}_{\mathbf{P}}$  on  $X$  in the manner we have explained before Prop. 4.9. This shows that  $X$  is uniquely endowed with an algebraic representation of  $\mathbf{P}_K$ . Consequently, there is a unique  $D(\mathfrak{g}, P)$ -module structure on  $M$  that extends its  $U(\mathfrak{g})$ -module structure and such that the action of  $U(\mathfrak{p})$ , as a subring of  $U(\mathfrak{g})$ , coincides with the action of  $U(\mathfrak{p})$  as a subring of  $D(P)$ . Moreover, any morphism  $M_1 \rightarrow M_2$  in  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  is automatically a homomorphism of  $D(\mathfrak{g}, P)$ -modules. In other words, we have a fully faithful embedding of categories

$$\mathcal{O}_{\text{alg}}^{\mathfrak{p}} \hookrightarrow \text{category of all } D(\mathfrak{g}, P)\text{-modules, finitely generated over } U(\mathfrak{g}) .$$

We now explain how one may compute the dimensions of the irreducible  $G$ -representations that occur in the image of  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  via the functor  $\mathcal{F}_P^G$ .

**Definition 5.5.** Let  $M$  be an object of the category  $\mathcal{O}$ . We call a standard parabolic subalgebra  $\mathfrak{p}$  *maximal for*  $M$  if  $M \in \mathcal{O}^{\mathfrak{p}}$  and if  $M \notin \mathcal{O}^{\mathfrak{q}}$  for all parabolic subalgebras  $\mathfrak{q}$  strictly containing  $\mathfrak{p}$ .

It follows from [14, sec. 9.4] that for every object  $M$  of  $\mathcal{O}$  there is unique standard parabolic subalgebra  $\mathfrak{p}$  which is maximal for  $M$ . The same definition applies for objects in the subcategory  $\mathcal{O}_{\text{alg}}$  in which case we also say that the standard parabolic subgroup  $P$  corresponding to  $\mathfrak{p}$  is *maximal for*  $M$ . In that case  $M$  lies in  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$ . We recall [26, Thm.5.3].

**Theorem 5.6.** *If the root system  $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$  has irreducible components of type  $B$ ,  $C$  or  $F_4$ , we assume  $p > 2$ , and if  $\Phi$  has irreducible components of type  $G_2$ , we assume that  $p > 3$ . Let  $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  be simple and assume that  $P$  is maximal for  $M$ . Then  $\mathcal{F}_P^G(M)'$  is a simple  $D(G_0)$ -module (and so, in particular, a simple  $D(G)$ -module).*

We let  $\lambda \in \mathfrak{t}_K^*$  be the differential of a locally analytic character of  $T$  and let  $P = P_I$  be adapted to  $\lambda$  in the sense of Example 5.4. Then  $\mathfrak{p}_K$  is maximal for  $M(\lambda)$ . Consider the simple quotient  $L(\lambda)$  and the coadmissible  $D(G)$ -module

$$\mathbf{L}_I(\lambda) := \mathcal{F}_P^G(L(\lambda))' .$$

We have  $\dim \mathbf{L}(\lambda) = \dim L(\lambda)$  by Thm. 4.11. Moreover, if  $\lambda \in X^*(\mathbf{T})$ , then  $L(\lambda)$  is an object of  $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$  and the  $D(G)$ -module  $\mathbf{L}(\lambda)$  is simple under the assumptions of the preceding

theorem. We therefore briefly recall the classical relation of  $\dim L(\lambda)$  to classical Goldie rank polynomials. To this end, we need to introduce some extra notation following [16, 2.7]. We let  $X^*(\mathbf{T}) \subseteq \Lambda$  be the integral weight lattice and let  $\Lambda^+$  and  $\Lambda^{++}$  be the subsets of dominant resp. strictly dominant weights. For simplicity we assume  $\lambda \in \Lambda$ . Recall that the isomorphism classes of the  $L(\mu)$ ,  $\mu \in \mathfrak{t}^*$  as well as the isomorphism classes of the  $M(\mu)$ ,  $\mu \in \mathfrak{t}^*$  form two different  $\mathbb{Z}$ -bases of the Grothendieck group of the abelian category  $\mathcal{O}$ , cf. [16, 4.5]. In particular, for any  $\mu \in \mathfrak{t}^*$

$$[L(\mu)] = \sum_{\mu' \in \mathfrak{t}^*} (L(\mu) : M(\mu')) [M(\mu')]$$

for some uniquely determined coefficients  $(L(\mu) : M(\mu')) \in \mathbb{Z}$ . For any  $\mu \in \Lambda^{++}$ , the number

$$a_\Lambda(w, w') := (L(w \cdot \mu) : M(w' \cdot \mu))$$

for  $w, w' \in W$  is independent of the choice of  $\mu$ , cf. [16, 4.14]. We fix once and for all an element  $t \in \mathfrak{t}$  such that  $\alpha(t) = 1$  for all  $\alpha \in \Delta$ . For fixed  $w \in W$  we let  $m = m_w \in \mathbb{N}_{\geq 0}$  be minimal such that

$$\tilde{f}_w^\Lambda := \frac{1}{m!} \sum_{w' \in W} a_\Lambda(w, w') (w'^{-1}(t))^m \in \text{Sym}^m(\mathfrak{t})$$

is nonzero, cf. [16, 9.13]. The number  $m_w$  does not depend on the particular choice of  $t \in \mathfrak{t}^*$ . In fact, different choices of  $t$  lead to polynomials that differ by a scalar in  $L^\times$ , cf. [16, 14.7]. The polynomial  $\tilde{f}_w^\Lambda$  is, up to scaling, the so-called *Goldie rank polynomial* of  $w \in W$ .<sup>3</sup>

We pick  $\mu \in \Lambda^+$  such that  $\lambda = w \cdot \mu = w(\mu + \rho) - \rho$  for some  $w \in W$ , put

$$(5.7) \quad S := B_\mu^0 := \{\alpha \in \Delta \mid \langle \mu + \rho, \alpha^\vee \rangle = 0\}$$

and let  $W_S$  be the subgroup of  $W$  generated by all  $s_\alpha$ ,  $\alpha \in S$ . Hence,  $W_S$  coincides with the stabilizer  $\{w' \in W : w' \cdot \mu = \mu\}$  according to [16, 2.5]. Let  $W^S$  be the unique system of representatives of maximal length for the left cosets in  $W/W_S$ . Since  $W \cdot \mu = W^S \cdot \mu$  we may and will assume that  $w \in W^S$ .

**Theorem 5.8.** *The module  $L(\lambda) = L(w \cdot \mu)$  has the dimension*

$$\dim_{U(\mathfrak{g})} L(w \cdot \mu) = \#\Phi^+ - m_w$$

---

<sup>3</sup>The polynomials  $\tilde{f}_w^\Lambda$  and their generalizations to arbitrary cosets  $\mathfrak{t}^*/\Lambda$  were introduced and studied by Joseph and build a bridge between primitive ideals of  $U(\mathfrak{g})$ , nilpotent adjoint orbits and the representation theory of  $W$ . For more details we refer to [16, Kap. 14].

where  $m_w$  denotes the degree of the polynomial  $\tilde{f}_w^\Lambda$ .

*Proof.* Since  $w \in W^S$ , we have  $B_\mu^0 = S \subset \tau_\Lambda(w)$  according to [16, 2.7(1)] (and in the notation of loc.cit.). We may therefore apply [16, Satz 9.12] to obtain  $\dim_{U(\mathfrak{g})} L(w \cdot \mu) = \#\Phi^+ - m_w$ . Note that all argument extend from the split semisimple case of loc.cit. to the more general split reductive case considered here and that Gelfand-Kirillov dimension may be replaced by canonical dimension.  $\square$

Remark: The wish to explicitly compute the polynomial  $\tilde{f}_w^\Lambda$  and its degree led to the formulation of the so-called Kazhdan-Lusztig conjecture [18]. This conjecture is now a theorem thanks to the work of Beilinson-Bernstein [3] and Brylinski-Kashiwara [6].

## 6. APPLICATION TO EQUIVARIANT LINE BUNDLES ON DRINFELD'S UPPER HALF SPACE

In this section we explain briefly how the results of the preceding sections combined with a theorem from [26] allow to compute the dimension of representations coming from line bundles on Drinfeld's half space.

We let  $\mathbf{G} = \mathrm{GL}_{d+1}$ . Moreover,  $\mathbf{B} \subset \mathrm{GL}_{d+1}$  equals the Borel subgroup of lower triangular matrices and  $\mathbf{T} \subset \mathbf{B}$  the diagonal torus. For a decomposition  $(n_1, \dots, n_s)$  of  $d+1$  the symbol  $\mathbf{P}_{n_1, \dots, n_s}$  denotes the corresponding lower standard parabolic subgroup of  $\mathrm{GL}_{d+1}$  with Levi subgroup  $\mathbf{L}_{n_1, \dots, n_s}$ .

Let  $\mathcal{X}$  be Drinfeld's half space of dimension  $d \geq 1$  over  $K$ . This is a rigid-analytic variety over  $K$  given by the complement of all  $K$ -rational hyperplanes in projective space  $\mathbb{P}_K^d$ , i.e.,

$$\mathcal{X} = \mathbb{P}_K^d \setminus \bigcup_{H \subsetneq K^{d+1}} \mathbb{P}(H) ,$$

where  $H$  runs over the set of  $K$ -rational hyperplanes in  $K^{d+1}$ . There is natural action of  $G = \mathrm{GL}_{d+1}(K)$  on  $\mathcal{X}$  induced by the algebraic action  $m : \mathbf{G} \times \mathbb{P}_K^d \rightarrow \mathbb{P}_K^d$  of  $\mathbf{G}$  defined by

$$g \cdot [q_0 : \dots : q_d] := m(g, [q_0 : \dots : q_d]) := [q_0 : \dots : q_d] g^{-1} .$$

Let  $s \in \mathbb{Z}$  and denote by  $\lambda' = (s, \dots, s) \in \mathbb{Z}^d$  the constant integral weight for  $\mathrm{GL}_d$ . Let  $r = \lambda_0 \in \mathbb{Z}$  and set

$$\lambda = (r, s, \dots, s) \in \mathbb{Z}^{d+1} .$$

We denote by  $\mathcal{L}_\lambda$  the homogeneous line bundle on  $\mathbb{P}_K^d = \mathrm{GL}_{d+1}/\mathbf{P}_{1,d}$  such that its fibre in the base point is the irreducible algebraic  $\mathbf{L}_{1,d}$ -representation corresponding to  $\lambda$ . Then we obtain  $\mathcal{L}_\lambda = \mathcal{O}(r-s)$  where the  $\mathbf{G}$ -linearization is given by the tensor product of the natural one on  $\mathcal{O}(r-s)$  with  $\det^s$ . The space of global sections  $H^0(\mathcal{X}, \mathcal{L}_\lambda)$  is a coadmissible  $D(G)$ -module. We may compute its dimension as follows.



Put  $w_j := s_j \cdots s_1$ , where  $s_i \in W$  is the (standard) simple reflection in the Weyl group  $W \cong S_{d+1}$  of  $G$ . Recall that  $\cdot$  denotes the dot action of  $W$  on  $X^*(\mathbf{T})_{\mathbb{Q}}$ . There is at most one integer  $0 \leq i_0 \leq d$ , such that  $H^{i_0}(\mathbb{P}_K^d, \mathcal{L}_{\lambda})$  is non-vanishing which is  $i_0 = 0$  for  $r \geq s$  resp.  $i_0 = d$  for  $s \geq r + d + 1$ . Otherwise, there is a unique integer  $i_0 < d$  with  $w_{i_0} \cdot \lambda = w_{i_0+1} \cdot \lambda$ . This is the case for  $0 \leq i_0 = s - r - 1 < d + 1$ . We put

$$\mu_{i,\lambda} := \left\{ \begin{array}{ll} w_{i-1} \cdot \lambda & : i \leq i_0 \\ w_i \cdot \lambda & : i > i_0 \end{array} \right\}, i = 1, \dots, d.$$

This is a  $\mathbf{L}_{(\mathbf{i}, \mathbf{d}-\mathbf{i}+1)}$ -dominant weight with respect to the Borel subgroup  $\mathbf{L}_{(\mathbf{i}, \mathbf{d}-\mathbf{i}+1)} \cap \mathbf{B}^+$  where  $\mathbf{B}^+$  denotes the upper triangular matrices in  $\mathrm{GL}_{d+1}$ . Consider the block matrix

$$z_j := \begin{pmatrix} 0 & I_j \\ I_{d+1-j} & 0 \end{pmatrix} \in G,$$

where  $I_k \in \mathrm{GL}_k(K)$  denotes the  $k \times k$ -identity matrix. We may regard  $z_j$  as an element of  $W$  and consider the weights  $z_j^{-1} \cdot \mu_{j,\lambda}$  for any  $j = 0, \dots, d-1$ . For each  $j$  we choose an element  $v_j \in W$  such that

$$v_j^{-1} \cdot (z_j^{-1} \cdot \mu_{j,\lambda}) \in \Lambda^+.$$

As explained in the discussion following (5.7), we can and will here assume additionally that  $v_j$  lies in the subset  $W^{S_j} \subset W$  corresponding to

$$S_j := B_{v_j^{-1} \cdot (z_j^{-1} \cdot \mu_{j,\lambda})}^0.$$

**Theorem 6.1.** *The coadmissible  $D(G)$ -module  $H^0(\mathcal{X}, \mathcal{L}_{\lambda})$  has the dimension*

$$\dim H^0(\mathcal{X}, \mathcal{L}_{\lambda}) = \#\Phi^+ - \min_{j=0, \dots, d-1} m_{v_j}.$$

Here,  $m_{v_j}$  denotes the degree of the polynomial  $\tilde{f}_{v_j}^{\Lambda}$ .

*Proof.* We abbreviate  $\mathcal{L}(\mathcal{X}) := H^0(\mathcal{X}, \mathcal{L}_{\lambda})$ . For  $j = 0, \dots, d-1$  the  $U(\mathfrak{g})$ -module  $L(z_j^{-1} \cdot \mu_{j,\lambda})$  lies in the category  $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}_{j+1,d-j}}$ . The parabolic  $\mathfrak{p}_{j+1,d-j}$  is maximal for it, cf. [26, Prop. 7.5], and so under the assumption of Thm. 5.6 the  $D(G)$ -module  $\mathcal{F}_{P_{(j+1,d-j)}}^G(L(z_j^{-1} \cdot \mu_{j,\lambda}))$  is simple, but we do not need this. In any case, there is a filtration of  $\mathcal{L}(\mathcal{X})$  by coadmissible  $D(G)$ -submodules

$$(6.2) \quad \mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X})^0 \supset \mathcal{L}(\mathcal{X})^1 \supset \dots \supset \mathcal{L}(\mathcal{X})^{d-1} \supset \mathcal{L}(\mathcal{X})^d = H^0(\mathbb{P}^d, \mathcal{L}),$$

such that for  $j = 0, \dots, d-1$ , there are  $D(G)$ -module extensions

$$(6.3) \quad 0 \rightarrow \mathcal{F}_{P_{(j+1,d-j)}}^G(L(z_j^{-1} \cdot \mu_{j,\lambda})) \rightarrow (\mathcal{L}(\mathcal{X})^j / \mathcal{L}(\mathcal{X})^{j+1}) \rightarrow \mathcal{Z}_j \rightarrow 0$$

where  $\mathcal{Z}_j$  is dual to a locally algebraic  $G$ -representation, cf. [26, Cor. 7.6] and [25, Cor. 2.2.9]. Since  $\mathcal{Z}_j$  has dimension zero, we obtain

$$\dim(\mathcal{L}(\mathcal{X})) = \max_{j=0,\dots,d-1} \dim \mathcal{F}_{P(j+1,d-j)}^G(L(z_j^{-1} \cdot \mu_{j,\lambda}))$$

and so Thm. 5.8 yields the assertion.  $\square$

## 7. DIMENSIONS IN THE UNITARY PRINCIPAL SERIES FOR $\mathrm{GL}_2(\mathbb{Q}_p)$

Let  $L = \mathbb{Q}_p$  in this section. We first discuss some preliminaries concerning locally analytic vectors in Banach space representations. In the second part we apply this to the unitary principal series for  $\mathrm{GL}_2(\mathbb{Q}_p)$  as introduced by Colmez [9].

We consider an arbitrary locally  $\mathbb{Q}_p$ -analytic group  $G$  of dimension  $d$ . Fix some arbitrary compact open subgroup  $H \subset G$  and let  $o_K[[H]]$  be the completed group ring of  $H$  with coefficients in  $o_K$  and put  $K[[H]] := K \otimes_{o_K} o_K[[H]]$ . Both rings are noetherian. The abelian category of admissible Banach space representations of  $G$  is anti-equivalent to the category of coadmissible modules, that is, finitely generated  $K[[H]]$ -modules equipped with a compatible linear action of  $G$  [33, Thm. 3.5]. If  $H' \subseteq H$  is another compact open subgroup, then the natural map  $K[[H']] \rightarrow K[[H]]$  is a finite (free) ring extension. The category of coadmissible modules is therefore independent of the choice of  $H$ . Applying [35, Lem. 8.8] to this latter ring extension shows  $j_{K[[H']]}(M) = j_{K[[H]]}(M)$  and one may therefore unambiguously define the dimension of a coadmissible module  $M$  to be

$$\dim M := d - j_{K[[H]]}(M) .$$

**Remark:** If  $H$  has no element of order  $p$ , then  $K[[H]]$  is an Auslander regular ring of global dimension  $d$  [37, Thm. 3.29].

The passage to locally analytic vectors  $V \mapsto V_{an}$  is an exact functor from admissible Banach space representations to admissible locally analytic representations [35, §7]. Let  $V'$  and  $(V_{an})'$  be the corresponding coadmissible modules.

**Proposition 7.1.** *One has  $\dim V' = \dim(V_{an})'$  for any admissible Banach space representation  $V$  of  $G$ .*

*Proof.* As representations of  $H$ , the functor  $V \mapsto V_{an}$  is given on the level of coadmissible modules as the base change  $M \mapsto D(H) \otimes_{K[[H]]} M$ . Since the homomorphism  $K[[H]] \rightarrow D(H)$  is faithfully flat [35, Thm. 5.2], the assertion follows from Lem. 2.2.  $\square$

We now specialize to certain reductive groups  $G$ . To this end, let  $\mathbf{G}$  be a connected split reductive group scheme over  $o_L$ . Let  $\bar{\kappa}$  be an algebraic closure of  $\kappa$ , the residue field of  $o_L$ . Let us consider the following three hypothesis on the geometric closed fibre  $\mathbf{G}_{\bar{s}} = \mathbf{G} \otimes_{o_L} \bar{\kappa}$  of  $\mathbf{G}$  which are familiar from the theory of modular Lie algebras (cf. [17], 6.3).

(H1) The derived group of  $\mathbf{G}_{\bar{s}}$  is (semisimple) simply connected.

- (H2) The prime  $p$  is good for the  $\bar{\kappa}$ -Lie algebra  $Lie(\mathbf{G}_{\bar{s}})$ .
- (H3) There exists a  $\mathbf{G}_{\bar{s}}$ -invariant non-degenerate bilinear form on  $Lie(\mathbf{G}_{\bar{s}})$ .

For example, the general linear group  $GL_n$  satisfies these conditions for all primes  $p$  (using the trace form for (H3)). Any almost simple and simply connected  $\mathbf{G}_{\bar{s}}$  satisfies these conditions if  $p \geq 7$  (and if  $p$  does not divide  $n + 1$  in case  $\mathbf{G}_{\bar{s}}$  is of type  $A_n$ ). For a more detailed discussion of these conditions we refer to loc.cit.

We assume from now on that (H1)-(H3) hold. As before,  $\Phi^+$  denotes a set of positive roots of  $\mathbf{G}$  and the number  $r$  denotes half the dimension of the minimal nilpotent coadjoint orbit of  $\mathbf{G}_{\bar{s}}$  ([7], Rem. 4.3.4). We let  $G$  from now be on a locally  $L$ -analytic group whose  $L$ -Lie algebra  $Lie(G)$  is isomorphic to  $\mathfrak{g}_L := L \otimes_{o_L} \mathfrak{g}$ . Let us identify these Lie algebras via such an isomorphism. Let  $d = \dim_L \mathfrak{g}_L$ .

**Proposition 7.2.** *Let  $V$  be an admissible  $G$ -Banach space representation. If  $V^{an}$  has an infinitesimal character, then  $\dim(V) \leq 2 \cdot \#\Phi^+$ . If  $\dim(V) \geq 1$ , then  $\dim(V) \geq r$ .*

*Proof.* The second statement follows from the main result of [2], extended to reductive groups satisfying (H1)-(H3) in [31]. Suppose now that  $V^{an}$  has an infinitesimal character. By [31, 9.4/9.6] it suffices to show that, for any  $n \geq 1$ , the dimension of a finitely generated module  $M$  over the Auslander regular ring  $\widehat{U(\mathfrak{g})}_{n,K}$  with infinitesimal character is bounded above by  $2 \cdot \#\Phi^+$ . Here,  $\widehat{U(\mathfrak{g})}_{n,K}$  denotes the  $\pi$ -adic completion (with subsequent inversion of  $\pi$ ) of the universal enveloping algebra  $U(\pi^n \mathfrak{g})$  for a choice of uniformizer  $\pi$  of  $o_L$ . We may choose a good double filtration for  $M$  and form its double graded module  $\text{Gr}(M)$  in the sense of [2, 3.2]. The latter is a finitely generated module over  $\text{Gr}(\widehat{U(\mathfrak{g})}_{n,K}) = \text{Sym}(\mathfrak{g}_{\kappa})$  whose support has dimension equal to the dimension of  $M$ . Since  $M$  has a central character,  $\text{Gr}(M)$  is annihilated by  $\text{Sym}(\mathfrak{g}_{\kappa})_+^{\mathbf{G}_k}$ , the ideal of invariant polynomials without constant term. Its support lies therefore in the cone of nilpotent elements of  $\mathfrak{g}_{\kappa}$  which has dimension  $2 \cdot \#\Phi^+$ .  $\square$

We recall that if  $V$  is absolutely irreducible, then  $V^{an}$  admits an infinitesimal character [11].

We now let  $\mathbf{G} = GL_2$  and turn to the unitary principal series of  $G = GL_2(\mathbb{Q}_p)$ . As usual,  $B \subset G$  denotes the Borel subgroup of upper triangular matrices. We fix a finite extension  $\mathbb{Q}_p \subseteq K$  as a coefficient field for the representations. We denote the continuous character  $x \mapsto x|x|$  of  $\mathbb{Q}_p^\times$  by  $\chi$ . Finally,  $\mathcal{G}_{\mathbb{Q}_p} = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  denotes the absolute Galois group of  $\mathbb{Q}_p$ .

In [10] Colmez establishes a correspondence

$$V \mapsto \Pi(V)$$

from absolutely irreducible 2-dimensional representations  $V$  of  $\mathcal{G}_{\mathbb{Q}_p}$  over  $K$  to absolutely irreducible unitary admissible  $G$ -representations. This correspondence is based on the

construction of a  $G$ -representation  $D(V) \boxtimes \mathbb{P}^1$  attached to  $V$  with central character  $\delta(x) = \chi^{-1} \det_V(x)$  where  $\det_V$  is the character of  $\mathbb{Q}_p^\times$  corresponding by local class field to the determinant of  $V$ . The representation  $D(V) \boxtimes \mathbb{P}^1$  is an extension of  $\Pi(V)$  by its dual twisted by  $\delta \circ \det$ . In particular, the central character of  $\Pi(V)$  equals  $\delta$  and Prop. 7.2 implies  $\dim \Pi(V) \leq 2$ . In the remainder of this section, we will determine  $\dim \Pi(V)$  in case  $\Pi(V)$  belongs to the unitary principal series, i.e. in case  $V$  is trianguline [9].

In the following, all  $(\varphi, \Gamma)$ -modules are taken over the classical Robba ring  $\mathcal{R}$ . Given a continuous character  $\eta : \mathbb{Q}_p^\times \rightarrow K^\times$ , the associated  $(\varphi, \Gamma)$ -module of rank 1 is denoted by  $\mathcal{R}(\eta)$ . Recall that a 2-dimensional Galois representation is called *trianguline*, if the associated étale  $(\varphi, \Gamma)$ -module is an extension of two (non-étales if  $V$  is irreducible) modules of rank 1. If  $\eta_1, \eta_2$  are two continuous characters  $\mathbb{Q}_p^\times \rightarrow K^\times$ , we denote the locally analytic induction  $\text{Ind}_B^G(\eta_2 \otimes \eta_1 \chi^{-1})$  simply by  $B^{an}(\eta_1, \eta_2)$  (note the reversed order of the  $\eta_i$ !).

**Proposition 7.3.** *One has  $\dim \Pi(V) = 1$  for any irreducible trianguline representation  $V$ .*

*Proof.* Let  $\Delta(s)$  be the étale  $(\varphi, \Gamma)$ -module associated with  $V$ . Here,  $s = (\delta_1, \delta_2, \mathcal{L})$  is the associated parameter consisting of continuous characters  $\delta_1, \delta_2 : \mathbb{Q}_p^\times \rightarrow K^\times$  and an element  $\mathcal{L} \in \mathbb{P}(\text{Ext}^1(\mathcal{R}(\delta_1), \mathcal{R}(\delta_2)))$ . In [8, Thm. 0.7] (compare also [23]) the locally analytic representation  $\Pi(V)^{an}$  is computed. Either we have the exact sequence of locally analytic  $G$ -representations

$$0 \rightarrow B^{an}(\delta_1, \delta_2) \rightarrow \Pi(V)^{an} \rightarrow B^{an}(\delta_2, \delta_1) \rightarrow 0$$

(the *generic* case) or we have an exact sequence of locally analytic  $G$ -representations

$$0 \rightarrow E_{\mathcal{L}} \rightarrow \Pi(V)^{an} \rightarrow B^{an}(\delta_2, \delta_1) \rightarrow 0$$

(the *special* case) where  $E_{\mathcal{L}}$  is an extension of a representation  $W(\delta_1, \delta_2)$  on a finite-dimensional  $K$ -vector space by  $\text{St}^{an}(\delta_1, \delta_2)$ . Here,  $W(\delta_1, \delta_2)$  is in fact a subrepresentation of  $B^{an}(\delta_1, \delta_2)$  and  $\text{St}^{an}(\delta_1, \delta_2)$  denotes the corresponding quotient of  $B^{an}(\delta_1, \delta_2)$ . We have  $\dim B^{an}(\eta_1, \eta_2) = 1$  according to Cor. 4.12 for any pair of continuous characters  $(\eta_1, \eta_2)$  which settles the generic case. Since  $\dim W(\delta_1, \delta_2) = 0$  we have  $\dim E_{\mathcal{L}} = 1$  and this settles the special case.  $\square$

The preceding proposition suggests the following question: Are there any absolutely irreducible 2-dimensional representations  $V$  of  $\mathcal{G}_{\mathbb{Q}_p}$  such that  $\dim \Pi(V) = 2$ ?

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